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Nearly Nonminimal Linear-Quadratic-Gaussian Compensators for Reduced-Order Control Design Initialization

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I. Introduction

THE development of linear-quadratic-Gaussian (LQG) theory was a major breakthrough in modern control theory since it provides a systematic way to synthesize high-performance controllers for nominal models of complex, multi-input multi-output systems. One of the well-known deficiencies of an LQG compensator, however, is that its minimal dimension is usually equal to the dimension of the design plant. This has led to the development of techniques to synthesize reduced-order approximations of the optimal full-order compensator (see Refs. 1–3 and the references therein).

The controller reduction methods almost always yield suboptimal (and sometimes destabilizing) reduced-order control laws since an optimal reduced-order controller is not usually a direct function of the parameters used to compute or describe the optimal full-order controller. Nevertheless, these methods are computationally inexpensive and sometimes do yield high performing and even nearly optimal control laws. An observation that holds true about most of these methods is that they tend to work best at low control authority.[2,3] To date, however, no algebraic conditions have been established that guarantee that a given suboptimal controller reduction method will work well at low authority.

This Note considers the balanced controller reduction algorithm of Ref. 1 and provides a constructive way of choosing the weights in an LQG control problem of dimension n such that for a given $n_c < n$ the corresponding n_cth-order controller obtained by this suboptimal reduction method has essentially the same performance as the LQG controller at low control authority. The usefulness of this result is for initializing homotopy algorithms for optimal reduced-order control design that requires a nearly optimal controller for an initial set of design parameters.[4]

The discussion here focuses on stable systems. It is shown that if the state weighting matrix R_1 or disturbance intensity V_1 has a specific structure in a basis in which the plant dynamics A is upper or lower block triangular, respectively, then at low control authority the corresponding LQG compensator is nearly nonminimal with minimal dimension n_c. It follows that the LQG compensator can be easily reduced to a n_cth-order controller having nearly the same performance. These results are directly applicable in initializing continuation and homotopy algorithms[4] that require a reduced-order controller that is nearly optimal for some set of initial design parameters corresponding to a low authority controller.

A special case of the conditions presented for R_1 and V_1 has a strong physical interpretation for structural control problems. In particular, assuming that all of the eigenvalues in the plant are complex (lightly damped structures) and n_c is an even number, then either R_1 is allowed to weight only $n_c/2$ structural modes or V_1 is allowed to disturb only $n_c/2$ structural modes.

II. Construction of Nearly Nonminimal LQG Compensators

Consider the nth-order linear time-invariant plant

$$\dot{x}(t) = Ax(t)+Bu(t)+D_1w(t), \qquad y(t) = Cx(t)+D_2w(t) \quad (1)$$

where (A, B) is stabilizable, (A, C) is detectable, $x \in R^n$, $u \in R^m$, $y \in R^l$, and $w \in R^d$ is a standard white noise disturbance with intensity I_d and rank $D_2 = l$. The intensities of $D_1 w(t)$ and $D_2 w(t)$ are thus given, by $V_1 \triangleq D_1 D_1^T \geq 0$ and $V_2 \triangleq D_2 D_2^T > 0$, respectively. Then, the LQG compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \qquad u(t) = -C_c x_c(t) \quad (2)$$

for the plant (1) minimizing the steady-state quadratic performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \frac{1}{t} E \int_0^t \left[x^T(s) R_1 x(s) + u^T(s) R_2 u(s)\right] ds \quad (3)$$

where $R_1 \geq 0$ and $R_2 > 0$ are the weighting matrices for the controlled states and controller input, respectively, the plant is given by

$$A_c = A - \Sigma P - Q\bar{\Sigma}, \qquad B_c = QC^T V_2^{-1}, \qquad C_c = R_2^{-1} B^T P \quad (4)$$

where $\Sigma \triangleq BR_2^{-1}B^T$ and $\bar{\Sigma} \triangleq C^T V_2^{-1} C$, and P and Q are the unique, nonnegative-definite solutions of

$$A^T P + PA + R_1 - P\Sigma P = 0 \quad (5)$$

and

$$AQ + QA^T + V_1 - Q\bar{\Sigma}Q = 0 \quad (6)$$

respectively. Furthermore, the shifted observability and controllability Gramians[1] of the compensator, $\hat{P}$ and $\hat{Q}$, are the unique, nonnegative-definite solutions of

$$(A - Q\bar{\Sigma})^T \hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P = 0 \quad (7)$$

and

$$(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q = 0 \quad (8)$$

respectively. The magnitudes of R_2 and V_2 relative to the state weighting matrix R_1 and plant disturbance intensity V_1 govern the regulator and estimator authorities, respectively. The selection of R_2 and V_2 such that $\|R_2\| \gg \|R_1\|$ or $\|V_2\| \gg \|V_1\|$ yields a low authority compensator. This section shows that when the open-loop plant is stable and (A, R_1) or (A, V_1) have a particular structure, the LQG controller approaches nonminimality as the controller authority decreases. To prove this result, we exploit structural properties of the solutions of Riccati and Lyapunov equations assuming the coefficient matrix A and the constant driving term R_1 have certain partitioned forms.

Lemma 2.1. Suppose

$$A = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} R_{1,1} & 0 \\ 0 & 0_{n-n_r} \end{bmatrix} \quad (9)$$

where $A_1, R_{1,1} \in R^{n_r \times n_r}$, $B_1 \in R^{n_r \times m}$, and $R_{1,1} > 0$, and assume A is asymptotically stable. Then the following statements hold.

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1) The unique, nonnegative-definite solution of the Riccati equation

$$A^T P + PA + R_1 - PBB^T P = 0 \tag{10}$$

is given by

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0_{n-n_r} \end{bmatrix} \tag{11}$$

where the $n_r \times n_r$ matrix P_1 is the unique, positive-definite solution of

$$A_1^T P_1 + P_1 A_1 + R_{1,1} - P_1 B_1 B_1^T P_1 = 0 \tag{12}$$

2) The unique, nonnegative-definite solution of the Lyapunov equation

$$A^T P + PA + R_1 = 0 \tag{13}$$

is given by

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0_{n-n_r} \end{bmatrix} \tag{14}$$

where the $n_r \times n_r$ matrix P_1 is the unique, positive-definite solution of

$$A_1^T P_1 + P_1 A_1 + R_{1,1} = 0 \tag{15}$$

Proof. The proof follows from Theorem 12.2 of Ref. 5. □

The following lemma states the dual of Lemma 2.1 if the coefficient matrix A is upper block triangular and V_1 is block diagonal.

Lemma 2.2. Suppose

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad C = [C_1 \quad C_2], \quad V_1 = \begin{bmatrix} V_{1,1} & 0 \\ 0 & 0_{n-n_r} \end{bmatrix} \tag{16}$$

where $A_1, V_{1,1} \in R^{n_r \times n_r}$, $C_1 \in R^{l \times n_r}$, and $V_{1,1} > 0$, and assume A is asymptotically stable. Then the following statements hold.

1) The unique, nonnegative-definite solution of the Riccati equation

$$AQ + QA^T + V_1 - QC^T CQ = 0 \tag{17}$$

is given by

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0_{n-n_r} \end{bmatrix}$$

where the $n_r \times n_r$ matrix Q_1 is the unique, positive-definite solution of

$$A_1 Q_1 + Q_1 A_1^T + V_{1,1} - Q_1 C_1^T C_1 Q_1 = 0 \tag{18}$$

2) The unique, nonnegative-definite solution of the Lyapunov equation

$$AQ + QA^T + V_1 = 0 \tag{19}$$

is given by

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0_{n-n_r} \end{bmatrix}$$

where the $n_r \times n_r$ matrix Q_1 is the unique, positive-definite solution of

$$A_1 Q_1 + Q_1 A_1^T + V_{1,1} = 0 \tag{20}$$

The following theorem shows that with the proper choice of the weighting and disturbance matrices, a low authority LQG controller for a stable plant is nearly nonminimal.

Theorem 2.1. Consider the plant given by Eq. (1) and assume A is asymptotically stable.

1) Suppose A and R_1 are partitioned as in Eq. (11) and let $V_2 \triangleq \beta \hat{V}_2$ where $\hat{V}_2$ is a positive-definite matrix and $\beta \in R$ is a positive scalar. Then for all $\delta > 0$, there exists N such that for all $\beta > N$, $(\lambda_{n_r+1}/\lambda_{n_r}) < \delta$, where λ_i represents the ith eigenvalue of $\hat{Q}\hat{P}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq \lambda_{i+1} \geq \cdots \geq 0$, and where $\hat{Q}$ and $\hat{P}$ are the shifted controllability and observability Gramians of the corresponding LQG compensator, satisfying Eqs. (8) and (7), respectively.

2) Suppose A and V_1 are partitioned as in Eq. (16) and let $R_2 \triangleq \alpha \hat{R}_2$, where $\hat{R}_2$ is a positive-definite matrix and $\alpha \in R$ is a positive scalar. Then for all $\delta > 0$, there exists N such that for all $\alpha > N$, $(\lambda_{n_r+1}/\lambda_{n_r}) < \delta$, where λ_i represents the ith eigenvalue of $\hat{Q}\hat{P}$.

Proof. 1) Partition

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix}$$

conformal to A in Eq. (9). A is asymptotically stable if and only if (A, B) and (A_1, B_1) are both stabilizable. Thus, it follows from property 1 of Lemma 2.1 that the unique, nonnegative-definite solution P of the Riccati equation (5) has the structure given by Eq. (11), which implies that

$$P\Sigma P = \begin{bmatrix} P_1 \Sigma_1 P_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

Next, using the partitioned structures in Eqs. (21) and (9) and the asymptotic stability of A, it follows from property 2 of Lemma 2.1 that there exists

$$\hat{P}_0 \triangleq \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & 0_{n-n_r} \end{bmatrix}$$

which is the unique, nonnegative-definite solution of

$$A^T \hat{P}_0 + \hat{P}_0 A + P\Sigma P = 0 \tag{22}$$

where the $n_r \times n_r$ matrix $\hat{P}_1$ is the unique, nonnegative-definite solution of $A_1^T \hat{P}_1 + \hat{P}_1 A_1 + P_1 \Sigma_1 P_1 = 0$. Next, computing Eq. (22)–Eq. (7) and using $V_2 = \beta \hat{V}_2$, yields

$$A^T \Delta\hat{P} + \Delta\hat{P} A + \beta^{-1}\left[(\bar{\Sigma}_0 Q\hat{P}) + (\bar{\Sigma}_0 Q\hat{P})^T\right] = 0 \tag{23}$$

where $\bar{\Sigma}_0 \triangleq C^T \hat{V}_2^{-1} C$ and $\Delta\hat{P} \triangleq \hat{P}_0 - \hat{P}$. Since A is asymptotically stable and Q and $\hat{P}$ satisfy Eqs. (6) and (7), respectively, Q and $\hat{P}$ are bounded for all β. Hence, it follows that $\hat{P} = \hat{P}_0 - \beta^{-1}\Delta\hat{P}_0$, where $\Delta\hat{P}_0$ is the solution of $A^T \Delta\hat{P}_0 + \Delta\hat{P}_0 A + [\bar{\Sigma}_0 Q\hat{P} + (\bar{\Sigma}_0 Q\hat{P})^T] = 0$. Now, rewriting Eq. (8) as $(A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + \beta^{-1} Q\bar{\Sigma}_0 Q = 0$, it follows that $\hat{Q} = \beta^{-1}\hat{Q}_0$, where $\hat{Q}_0$ satisfies $(A - \Sigma P)\hat{Q}_0 + \hat{Q}_0(A - \Sigma P)^T + Q\bar{\Sigma}_0 Q = 0$. Next, using $\hat{P} = \hat{P}_0 - \beta^{-1}\Delta\hat{P}_0$ and $\hat{Q} = \beta^{-1}\hat{Q}_0$, we obtain (for large β)

$$S \triangleq \hat{Q}\hat{P} = \begin{bmatrix} \beta^{-1} S_1 & \beta^{-1} S_{12} \\ \beta^{-2} S_{21} & \beta^{-2} S_2 \end{bmatrix}$$

where $S_1 \in R^{n_r \times n_r}$, $S_2 \in R^{n-n_r \times n-n_r}$, and S_1 is nonsingular. Note that since $\hat{Q}$ and $\hat{P}$ are nonnegative definite, S is semisimple, and the eigenvalues of S are real and nonnegative. Hence, the ratio of the eigenvalues of S is the same as the corresponding ratio of the eigenvalues of βS. Next, define

$$S' \triangleq \begin{bmatrix} S_1 & S_{12} \\ 0 & 0 \end{bmatrix}$$

and note that $\lim_{\beta\to\infty} \beta S = S'$. Furthermore, note that the eigenvalues of S' are the collection of n_r eigenvalues of S_1 along with $(n - n_r)$ zero eigenvalues. Now, since the eigenvalues of a matrix are continuous with respect to the parameters of the matrix, it follows that for all $\epsilon > 0$, there exists N such that for all $\beta > N$, $\lambda_{S_1,i} - \epsilon < \lambda_{\beta S,i} < \lambda_{S_1,i} + \epsilon$, $i = 1, \ldots, n_r$, and $\lambda_{\beta S,i} < \epsilon$, $i = n_r + 1, \ldots, n$, where $\lambda_{\beta S,i}$ and $\lambda_{S_1,i}$ represent the ith eigenvalue of βS and S_1, respectively, in descending order. Hence, it follows that for all $\delta > 0$, there exists N such that for all $\beta > N$, $(\lambda_{S,n_r+1})/\lambda_{S,n_r} < \delta$. The proof of 2 is dual to the proof of 1. □

Remark 2.1. Theorem 2.1 provides two constructions for weighting matrix selection resulting in a nearly nonminimal, low authority LQG compensator for a stable plant. The first approach involves the transformation of the plant A into coordinates such that A has the representation given by Eq. (9). In this case, the weighting matrix R_1 is selected as in Eq. (9) and with rank $R_1 = n_r$. By decreasing the authority of the compensator or, equivalently, increasing $\|V_2\|$ or β, the eigenvalue ratio, $(\lambda_{n_r+1})/\lambda_{n_r}$ of the LQG compensator decreases and the LQG compensator approaches nonminimality with minimal dimension of n_r. Using a dual approach, with A and V_1 partitioned as in Eq. (16), by increasing $\|R_2\|$ or α, the resulting LQG compensator approaches nonminimality. In the limiting case, however, as $\alpha \to \infty$ or $\beta \to \infty$ then it follows from Eqs. (7) and (8) that $\hat{P} \to 0$ and $\hat{Q} \to 0$, respectively.

Remark 2.2. Note that if A is in modal form, then Eqs. (9) and (16) are automatically satisfied. In this case R_1 given by Eq. (9) describes a state weighting matrix in which only the states pertaining to selected modes are weighted. Similarly, V_1 given by Eq. (16) describes a disturbance that excites only certain modes. It is not uncommon for these conditions to be satisfied or nearly satisfied in practice.

III. Numerical Illustrative Example

To illustrate the proper choices of the weighting matrices resulting in a nearly nonminimal, low authority LQG compensator for a stable continuous-time plant, consider a simply supported beam with two collocated sensor/actuator pairs.[6] Assuming the beam has length 2 and that the sensor/actuator pairs are placed at coordinates $a = 55/172$ and $b = 46/43$, a five-mode continuous-time model, along with the problem data is given in Ref. 6. The noise intensities are $V_1 \triangleq 0.1 I_{10}$ and $V_2 = \beta I_2$. The design objective is to minimize the continuous-time cost (3) with $R_2 = \alpha I_2$. Note that the magnitude of the positive real numbers α and β are the indicators of the controller authority level. For this system, A has the representation given by Eqs. (9) and (16) with $A_{12} = 0$ and $A_{21} = 0$, respectively. Here, we illustrate the results of property 1 of Theorem 2.1 for the case of $n_r = 2$. Setting $\alpha = 0.1$, by selecting the weighting matrix

$$R_1 = \begin{bmatrix} I_{n_r} & 0 \\ 0 & 0 \end{bmatrix}$$

and increasing β (hence, decreasing the compensator authority), the resulting LQG compensator approaches nonminimality with minimal dimension of n_r or, equivalently, $[\lambda_{n_r+1}(\hat{Q}\hat{P})]/[\lambda_{n_r}(\hat{Q}\hat{P})] \to 0$ where λ_i is the sorted (in descending order) ith eigenvalue of $\hat{Q}\hat{P}$. Figure 1 shows the ratio curve for $n_r = 2$ with $\beta \in (0.01, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5, 10^6)$. The curve clearly indicates that the ratio decreases as β increases. To illustrate that suboptimal controller reduction methods yield nearly optimal reduced-order compensators for low authority control problems, Fig. 1 also shows the norm of the cost gradient of the second-order controller obtained by balancing. The cost gradient is defined as

$$\left[\left(\mathrm{vec}\frac{\partial J}{\partial A_c}\right)^T \left(\mathrm{vec}\frac{\partial J}{\partial B_c}\right)^T \left(\mathrm{vec}\frac{\partial J}{\partial C_c}\right)^T\right]^T$$

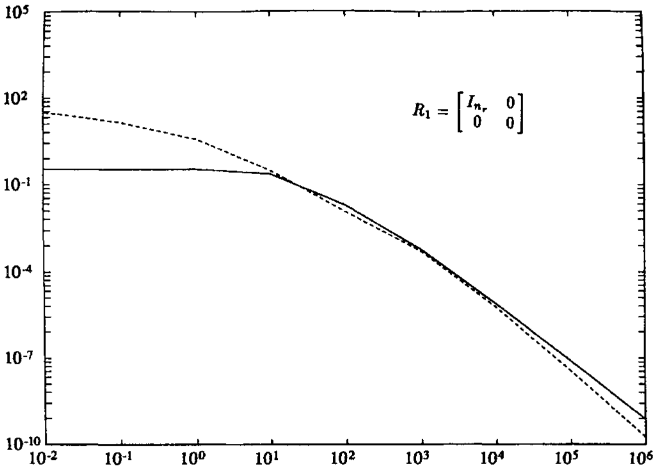

Fig. 1 Ratio curve $\{[\lambda_{n_r+1}(\hat{Q}\hat{P})]/[\lambda_{n_r}(\hat{Q}\hat{P})]\}$ of the LQG controller (——) and the norm of the cost gradient of the second-order balanced controller (- - - -) vs control authority (β) for $n_r = 2$.

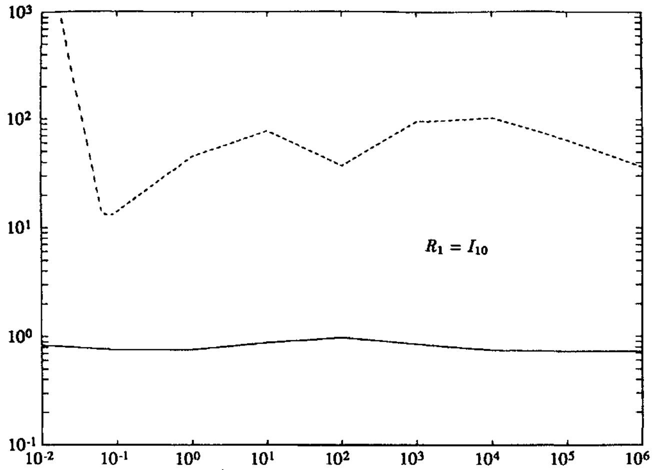

Fig. 2 Ratio curve $\{[\lambda_{n_r+1}(\hat{Q}\hat{P})]/[\lambda_{n_r}(\hat{Q}\hat{P})]\}$ of the LQG controller (——) and the norm of the cost gradient of the second-order balanced controller (- - - -) vs control authority (β) for $n_r = 2$.

where vec($\cdot$) denotes the column stacking operator. The cost gradient curve indicates the balanced controller approaches the optimal reduced-order compensator as β increases, or as the control authority decreases.

Conversely, if the weighting term R_1 for the preceding example does not have the structure given by Eq. (9), decreasing the controller authority (i.e., increasing β) may not necessarily yield a nearly nonminimal LQG compensator. As an apparent consequence, the norm of the cost gradient of the corresponding second-order balanced controller does not approach zero as the control authority decreases. This is illustrated in Fig. 2 for $n_r = 2$ and $R_1 = I_{10}$. Note that for this particular example, at $\beta = 0.01$ the balanced controller destabilizes the closed-loop system and, hence, the norm of the cost gradient becomes infinite.

IV. Conclusion

By exploiting structural properties of the solutions of the regulator and observer LQG Riccati equations this paper shows that, for continuous-time stable systems, if the coefficient matrix A and driving weighting term R_1 (or V_1) have specific structures, the corresponding LQG compensator becomes nonminimal as the control authority is decreased. The results are directly applicable for initializing continuation and homotopy algorithms that require a reduced-order controller that is nearly optimal for some set of initial design constraints.

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